Econometrics 1 (ECON 4003)
Suggested Solutions - Tutorial 3

Max Schroder

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Abstract

This guide is supposed to be complementary to the official solutions supplied by the lecturer. All errors are my own.

Question 1

This question tells you that the fund return X follows a **normal distribution** with a specific mean and variance. Specifically $X \sim N(0.05, 0.04^2)$.

a)

So we are looking for the probability that X turns out to be less than 0. Or differently, we are trying to measure the area under the **probability density function** ranging from $-\infty$ to 0.[1] The important step that we have to remember here is that we can transform any random variable that has been drawn from a normal distribution into a random variable drawn from a **standard normal** distribution: $Z \sim N(0, 1)$. This is possible, because a normal distribution is completely defined by it's first two *moments,* namely the *mean* and the *variance.* In previous tutorials we have seen how to manipulate these two variables using addition and multiplication. Effectively it is possible to "shift" and "stretch" any given normal distribution into the shape of a **standard normal**. The formula to achieve this is given by $Z = \frac{X-\mu_x}{\sigma_x}$ this essentially means that we are "shifting" the center of the pdf of X as well as stretching it out so that it becomes a standard normal. Remember that when you are applying this formula you are effectively manipulating two sides of an inequality:

$$P(X < 0) = P(\frac{X - \mu_x}{\sigma_x} < \frac{0 - \mu_x}{\sigma_x}) = P(Z < \frac{0 - \mu_x}{\sigma_x}) \quad (1)$$

This is relevant since we know the area covered by a standard normal distribution - you can look it up in the statistical tables and thus find the relevant probability.

[1] If you are looking for a bit more information I suggest https://www.youtube.com/watch?v=Fvi9A_tEmXQ and related resources.

b)

Same as before, only now we are trying to measure the area **above** 0.15. Here it's good to remember that the area below a certain point and the area above that point must sum to 1.[2] Hence $\Pr(X > 0.15) = 1 - \Pr(X \leq 0.15)$. Note that by convention, the cutoff point that you choose is always included in the lower probability set, hence the *"smaller than or equal"* sign.[3] This is also useful since sometimes the producers of statistical tables are lazy and only give you the lower half of the pdf and you will have to use the symmetry of the normal distribution to find the relevant probabilities.

c)

Here you repeat the last two exercises with new values for μ_x and σ_x. Do you see how the distribution is now a lot more spread out with more probability mass in the tails?

Question 2

This is another application of something that we did last week. The only thing to not get confused about is that mean and expected return mean the same thing here. Also note that the sum of two (or any number) of normally distributed variables is also normally distributed.[4] This is one of the nice properties of the normal distribution.

a)

We are looking for: $E(P) = E(0.25R_A + 0.75R_B)$ using our rules for the expectations operator we can rewrite this as: $E(P) = 0.25E(R_A) + 0.75E(R_B)$ which gives the solution.

b)

Remember that $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$. Now realize that $X = 0.25R_A$ and $Y = 0.75R_B$ and substitute into the formula. Note that you can treat the Covariance as a linear operator, i.e. $Cov(aX, bY) = a * b * Cov(X, Y)$. The most important part here is that you have to remember that you have been given the **Correlation** coefficient and not the value of the **Covariance**.

[2] Do you see why?

[3] Technically this doesn't matter however, since any specific point on the pdf has a probability of exactly 0.

[4] Such sums are sometimes referred to as "mixtures of normals".

c)

This one is a little easier since we have been told that $Cov(R_A, R_B) = 0$. As you can see, the standard deviation of the overall portfolio returns has decreased[5]. Bonus question: is there a way to reduce the variance of the portfolio return even more?

Question 3

This question deals with estimators and their properties. Estimators are essentially transformations of the data that are supposed to help us recover the *deep (structural)* parameters that underlie the random process that we are investigating. A good estimator should be **unbiased** - i.e. on average give us the correct value of the parameter that we are looking for; as well as **efficient** - i.e. the unbiased estimator has the smallest possible sampling error.[6] Sampling error comes from the fact that we are usually only observing a small *sample* of the whole universe of possible outcomes and therefore cannot trust even unbiased estimators to give us the correct answer every time. Randomness will assert itself and distort our *estimate* away from the the true *parameter*. The best we can hope for is to have an estimator that is close to the truth.

The axiomatic study of the properties of estimators is sometimes called "asymptotics". In recent years, some researchers have abandoned the asymptotic approach for confirming the properties of their (arguably way to complicated) estimators and instead rely on testing those using large scale computer simulations. This is known as "Monte Carlo".

a)

Here we just apply the rules of the linear expectations operator.

b)

For this question I'd assume that the draws from the distribution of Y are **i.i.d.** (*independent and identically distributed*) and hence the correlation between all outcomes is 0. And thank god for that, since otherwise the covariance term in the sum would be a nightmare. If you are a little confused about how to apply the variance formula in the case of 3 variables, try this conceptual trick: First define a new variable $Y_{1\&2} = \frac{1}{2}Y_1 + \frac{1}{3}Y_2$, then calculate $Var(Y_{1\&2})$ and then calculate $Var(\tilde{Y})$ as $Var(Y_{1\&2} + \frac{1}{6}Y_3)$.

[5] do you see why?

[6] Remember that a transformation of a random variable is itself a random variable.

c)

Here we are calculating

$$Var(\bar{Y}) = Var(\frac{1}{3}Y_1 + \frac{1}{3}Y_2 + \frac{1}{3}Y_3) = \left(\frac{1}{3}\right)^2 \sum_i Var(Y_i) = \frac{1}{3}\sigma^2 \qquad (2)$$

. As it turns out, the mean has a smaller sampling variance than the $\tilde{Y}$ estimator. Since the mean is also an unbiased estimator[7] we can say that it is a *more efficient* estimator. Often it is the case, that there exists an unbiased estimator of some parameter, that also has the smallest sampling variance. However in some more complicated problems this is not always the case and we will have to make trade-offs between unbiasedness and efficiency.

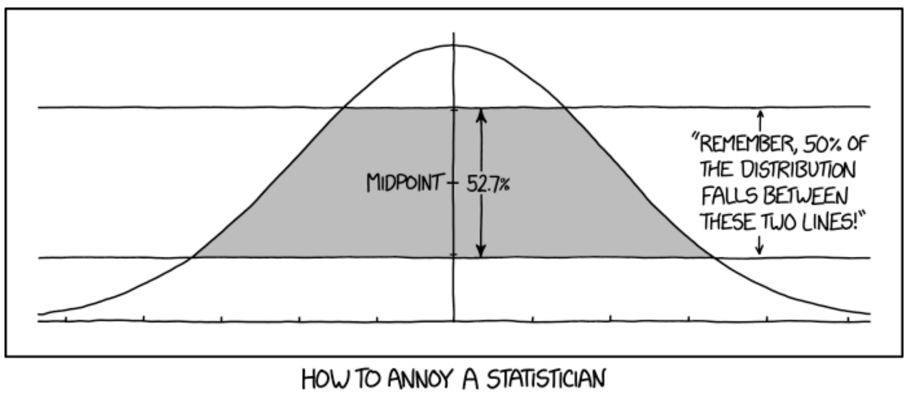

Figure 1: Source: https://xkcd.com

[7] Can you confirm this?