Economics 1B
Suggested Solutions - Seminar 5

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Abstract

This guide is supposed to be complementary to the official solutions supplied by the lecturer. All errors are my own.[1]

1 Review of some Matrix Algebra

Determinant:

The determinant is a scalar value that can be calculated from a square matrix, and that summarizes some properties of the matrix. For our purposes, the determinant will be most useful for calculating the inverse of a given square matrix. The simplest case of a 2×2 matrix: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix, then the **determinant** is given by $|A| = ad - bc$.

At higher orders determinante calculations quickly become more cumbersome (at least using standard methods). For example for a 3×3 matrix:

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|B| = a \left| \begin{bmatrix} & e & f \\ & h & i \end{bmatrix} \right| - b \left| \begin{bmatrix} d & & f \\ g & & i \end{bmatrix} \right| + c \left| \begin{bmatrix} d & e & \\ g & h & \end{bmatrix} \right|$$

$$|B| = aei + bfg + cdh - ceg - bdi - afh$$

This process essentially proceeds by decomposing the problem of finding the determinant of a large matrix into finding the determinants of several smaller matrizes. The determinante of each of the submatrizes that is created by deleting row i and column j of the original matrix, is called the **minor** (M_{ij}) of B. For this reason, this procedure is called the "minor expansion formular".

Inverse of a Matrix:

[1] Many thanks to Johanna for sharing her solutions with me on this occasion.

The inverse of a square matrix A is another square matrix A^{-1} such that $AA^{-1} = I$ where I is the identity matrix. Importantly, not all (square) matrices are invertible. Such matrices are called *singular.* A matrix is singular **iff** its determinant is zero. For a (non singular) 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the inverse is provided by: $A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Eigenvalues & Eigenvectors:

Eigenvalues and Eigenvectors summarize another set of characterstics of a matrix. Generally, they are associated with the direction of the space spanned by the matrix, but in this context we will mainly use them to find solutions to systems of linear equations.

If M is a square $m \times m$ matrix, then the eigenvalues λ and associated eigenvectors ν can be calculated as the solution to:

$$\begin{aligned} M\nu &= \lambda v \\ (M - \lambda I)\,\nu &= 0 \end{aligned}$$

To find the eigenvalues and vectors we first solve $|M - \lambda I| = 0$. This determinant is sometimes called the "characterisitic polynominal" of M and the eigenvalues are roots of this equation.

For example, let $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then

$$\begin{aligned} |M - \lambda I| &= \\ \left| \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| &= \\ \left| \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \right| &= \\ & \quad 3 - 4\lambda + \lambda^2 \end{aligned}$$

This polynominal has two solutions: $\lambda_1 = 1$ and $\lambda_2 = 3$. Now that we found the eigenvalues, we can find the eigenvectors in each case:

For $\lambda_1 = 1$ we have:

$$(M - I)\,\nu_1 = 0$$

$$\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_{11} + v_{12} = 0$$
$$v_{11} + v_{12} = 0$$

$$v_{11} = -v_{12}$$

$$\nu_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda_1 = 3$ we have:

$$(M - 3I)\,\nu_2 = 0$$
$$\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-v_{21} + v_{22} = 0$$
$$v_{21} - v_{22} = 0$$
$$v_{21} = v_{22}$$
$$\nu_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note that the eigenvectors v_1 and v_2 are defined up to multiplication with a scalar. So $\tilde{v}_1 = av_1$ is also an eigenvector associated with the eigenvalue 1.

2 Question 7.2

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix} \; B = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$$

2.1 1.1a)

$$|A| = 2 * 1 - 5 * 1 = -3$$

2.2 1.1b)

$$|B| = 1 * 4 - 2 * 0 = 4$$

2.3 1.1c)

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 7 & 4 \end{bmatrix} \\ |AB| &= 4 * 4 - 7 * 4 = -12 \end{aligned}$$

2.4 1.1d)

$$|A| \times |B| = |AB|$$

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix} \ B = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$$

2.5 1.2a)

$$A^{-1} = -\frac{1}{3} \begin{bmatrix} 1 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{5}{3} & -\frac{2}{3} \end{bmatrix}$$

2.6 1.2b)

$$B^{-1} = \frac{1}{4} \begin{bmatrix} 4 & -0 \\ -2 & 1 \end{bmatrix} \ = \begin{bmatrix} 1 & -0 \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

2.7 1.2c)

$$(AB)^{-1} = -\frac{1}{12} \begin{bmatrix} 4 & -4 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{7}{12} & -\frac{1}{3} \end{bmatrix}$$

3 1.2d)

$$(AB)^{-1} = B^{-1}A^{-1}$$

4 Question 2

If $\begin{bmatrix} 2 & -1 \\ 3 & a \end{bmatrix}$ and $\begin{bmatrix} 2 & b \\ 3 & -4 \end{bmatrix}$ are singular, then the determinant in each case needs to be zero. Hence we need to solve: $2a + 3 = 0$ and $-8 - 3b = 0$.

$$\begin{aligned} a &= -\frac{3}{2} \\ b &= -\frac{8}{3} \end{aligned}$$

4.1 Question 5

The demand and supply functions for two independent goods are given:

$$\begin{aligned}
Q_{D1} &= 50 - 2p_1 + p_2 \\
Q_{D2} &= 10 + p_1 - 4p_2 \\
Q_{S1} &= -20 + p_1 \\
Q_{S2} &= -10 + 5p_1
\end{aligned}$$

4.2 a)

Show that the equilibrium prices satisfy:

$$\begin{bmatrix} 3 & -1 \\ -1 & 9 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 70 \\ 20 \end{bmatrix}$$

First, in equilibrium the quantity demanded, must be equal to the quantity supplied for each good:

$$\begin{aligned}
Q_{D1} &= Q_{S1} \\
Q_{D2} &= Q_{S2}
\end{aligned}$$

rearanging gives:

$$\begin{aligned}
50 - 2p_1 + p_2 &= -20 + p_1 \\
\mathbf{3p_1 - p_2} &= \mathbf{70}
\end{aligned}$$

and:

$$\begin{aligned}
10 + p_1 - 4p_2 &= -10 + 5p_1 \\
\mathbf{-p_1 + 9p_2} &= \mathbf{20}
\end{aligned}$$

which constitute the system that was required.

4.3 b)

find the inverse of the 2x2 matrix in part **a)** and hence find the equilibrium prices.

Note that the system has the following form:

$$Ax = b$$

as we previewed in the last tutorial, we can use matrix algebra to solve for x :

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

Here we know $A = \begin{bmatrix} 3 & -1 \\ -1 & 9 \end{bmatrix}$, $b = \begin{bmatrix} 70 \\ 20 \end{bmatrix}$ and are looking for the vector of unknowns $p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$.

$$A^{-1} = \frac{1}{26} \begin{bmatrix} 9 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\begin{aligned} A^{-1}b &= p \\ \frac{1}{26} \begin{bmatrix} 9 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 70 \\ 20 \end{bmatrix} &= \begin{bmatrix} 25 \\ 5 \end{bmatrix} \end{aligned}$$

4.4 7.2*Question 8

Find the determinant of the matrix

$$A = \left| \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & a \\ 3 & 1 & 4 \end{bmatrix} \right|$$

I am not going through all the steps here (you can try it as an exercise), but $|A| = \frac{1}{a-1}$.Now using some mathematical commonsense, we can deduce that A is non singular if $a \neq 1$.

Again, inverting a 3×3 is a pain.[2]

$$A^{-1} = \frac{1}{a-1} \begin{bmatrix} -a & -1 & a \\ 3a-4 & -1 & 3-2a \\ 1 & 1 & -1 \end{bmatrix}$$

[2] If you want a recap have a look at: https://www.khanacademy.org/math/algebra-home/alg-matrices/alg-determinants-and-inverses-of-large-matrices/v/inverting-3x3-part-1-calculating-matrix-of-minors-and-cofactor-matrix

4.5 7.3 Question 2

Cramers Rule says that for a system $Ax = b$ we can find the value of the ith element of x by:

$$x_i = \frac{|A_i|}{|A|}$$

where A_i is the matrix that is created by replacing the ith column of A with b.

Use Cramer's rule to find the value of y which satisifies each of the following pairs of simultaneous equations:

4.6 a)

$$\begin{aligned} x + 3y &= 9 \\ 2x - 4y &= -2 \end{aligned}$$

$$x = \frac{\begin{vmatrix} 9 & 3 \\ -2 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix}} = \frac{[9 \times -4] - [-2 \times 3]}{[1 \times -4] - [2 \times 3]} = \frac{-30}{-10} = 3$$

$$y = \frac{\begin{vmatrix} 1 & 9 \\ 2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix}} = \frac{[1 \times -2] - [2 \times 9]}{[1 \times -4] - [2 \times 3]} = \frac{-20}{-10} = 2$$

4.7 b)

$$\begin{aligned} 5x - 2y &= 7 \\ 2x + 3y &= -1 \end{aligned}$$

$$x = \frac{\begin{vmatrix} 7 & -2 \\ -1 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix}} = 1$$

$$y = \frac{\begin{vmatrix} 5 & 7 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix}} = -1$$

4.8 c)

$$\begin{aligned} 2x + 3y &= 7 \\ 3x - 5y &= 1 \end{aligned}$$

$$x = \frac{\begin{vmatrix} 7 & 3 \\ 1 & \text{-5} \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & \text{-5} \end{vmatrix}} = 2$$

$$y = \frac{\begin{vmatrix} 2 & 7 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & \text{-5} \end{vmatrix}} = 1$$

5 Question 5

5.1 Part (1)

$$\begin{aligned} Y &= C + I^* \\ C &= aY + b \end{aligned}$$

5.2 a)

Express this system in the form $Ax = b$, where $x = \begin{bmatrix} Y \\ C \end{bmatrix}$ and A and b are 2×2 and 2×1 matrices to be stated.

$$\begin{bmatrix} 1 & -1 \\ -a & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I^* \\ b \end{bmatrix}$$

5.3 b)

Use Cramer's rule to solve this system for C:

$$C = \frac{\begin{vmatrix} 1 & I^* \\ -a & b \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -a & 1 \end{vmatrix}} = \frac{b + aI^*}{1 - a}$$

6 Question 7.3*

6.1 Part (b)

Use the Cramers Rule to solve

$$\begin{bmatrix} 3 & 2 & -2 \\ 4 & 3 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -15 \\ 17 \\ -1 \end{bmatrix}$$

for x_2

$$x_2 = \frac{|A_2|}{|A|} = \frac{126}{42} = 3$$